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SHORTER COMMUNICATIONS

APPROXIMATIONS FOR CONDUCTION WITH FREEZING OR MELTING

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NOMENCLATURE

- a, radius of cylinder or sphere;
- c, heat capacity;
- j, heat flux density;
- k, thermal conductivity;
- L, latent heat;
- T, temperature;
- x, distance from surface at T_w;
- z, distance from corner at T_w in plane of symmetry.

Greek symbols

- α, thermal diffusivity;
- λ, root of equation (3);
- ρ, density.

Subscripts

- e, effective value;
- f, at freezing front;
- j, based on heat flux density;
- l, in unfrozen region;
- 0, initial;
- s, in frozen region;
- w, at surface;
- x, based on freezing-front location.

INTRODUCTION

HEAT conduction with freezing or melting has been studied extensively in the 85 years since Stefan [1] considered the problem in connection with the thickness of the polar ice. Current applications of interest include the storage of energy, the underground storage of cryogenic fluids, construction and operations in the permafrost region, the casting of metals and the preservation of food and biological materials. Bankoff [2], Mori and Araki [3], Gupta and Churchill [4] and others have reviewed many of these investigations and only the directly relevant will be noted herein.

Neumann is credited by Carslaw and Jaeger ([5], p. 282) with the simple exact solution, given below, for phase-front motion parallel to a flat surface. Because of mathematical difficulties arising from the non-linearity associated with the movement of the phase-front, subsequent solutions are for highly idealized conditions, are approximate or are very complex in form. Even the numerical solutions have generally been restricted to one dimension. Most of the approximate methods suffer from complexity, inapplicability to two and three dimensions, inaccuracy or undefined accuracy. Hence, a simple, approximate method which is applicable for all boundary conditions and geometries, including those of two and three dimensions, still appears to be needed. The objective of this paper is to describe such a model.

DEVELOPMENT

Neumann solution

The model is based on an examination of the Neumann solution ([5], p. 285) for freezing of material in the half-space, $x > 0$, at a uniform initial temperature, T_0, above the freezing point, T_f, by the sudden application and maintenance of a sub-freezing temperature, T_w, at the surface. Heat transfer is postulated to take place by conduction only with different but constant physical properties in the two phases. The motion due to the change in density across the freezing-front is neglected. The solution can be written as

$$\frac{T-T_w}{T_f-T_w} = \frac{\operatorname{erf}\{x/2(\alpha_s t)^{1/2}\}}{\operatorname{erf}\{\lambda\}}, \quad x < x_f \tag{1}$$

and

$$\frac{T_0-T}{T_0-T_f} = \frac{1-\operatorname{erf}\{x/2(\alpha_l t)^{1/2}\}}{1-\operatorname{erf}\{\lambda(\alpha_s/\alpha_l)^{1/2}\}}, \quad x > x_f \tag{2}$$

where λ is the root of

$$\frac{e^{-\lambda^2}}{\lambda \operatorname{erf}\{\lambda\}} - \left(\frac{T_0-T_f}{T_f-T_w}\right)\left[\frac{(k\rho c)_l}{(k\rho c)_s}\right]^{1/2} \frac{e^{-\lambda^2\alpha_s/\alpha_l}}{\lambda(1-\operatorname{erf}\{\lambda[\alpha_s/\alpha_l]^{1/2}\})} = \frac{\pi L}{c_s(T_f-T_w)}. \tag{3}$$

It follows from equation (1) that

$$x_f = 2\lambda(\alpha_s t)^{1/2} \tag{4}$$

and

$$j_w = k_s(T_f-T_w)/(\pi\alpha_s t)^{1/2} \operatorname{erf}\{\lambda\}. \tag{5}$$

Values of λ and $\operatorname{erf}\lambda$ have been tabulated and plotted for a wide range of the parameters $L/c_s(T_f-T_w)$, $(T_0-T_f/T_f-T_w)[(k\rho c)_l/(k\rho c)_s]^{1/2}$ and α_s/α_l by Churchill and Evans [6].

Zero latent heat

For the limiting case of zero latent heat, and uniform (unsubscripted) properties, equations (1) and (2) reduce to

$$(T-T_w)/(T_0-T_w) = \operatorname{erf}\{x/2(\alpha t)^{1/2}\} \tag{6}$$

and equation (5) to

$$j_w = k(T_0-T_w)/(\pi\alpha t)^{1/2}. \tag{7}$$

For the T_f isotherm

$$x_f = 2(\alpha t)^{1/2} \operatorname{erf}^{-1}\{(T_f-T_w)/(T_0-T_w)\}. \tag{8}$$

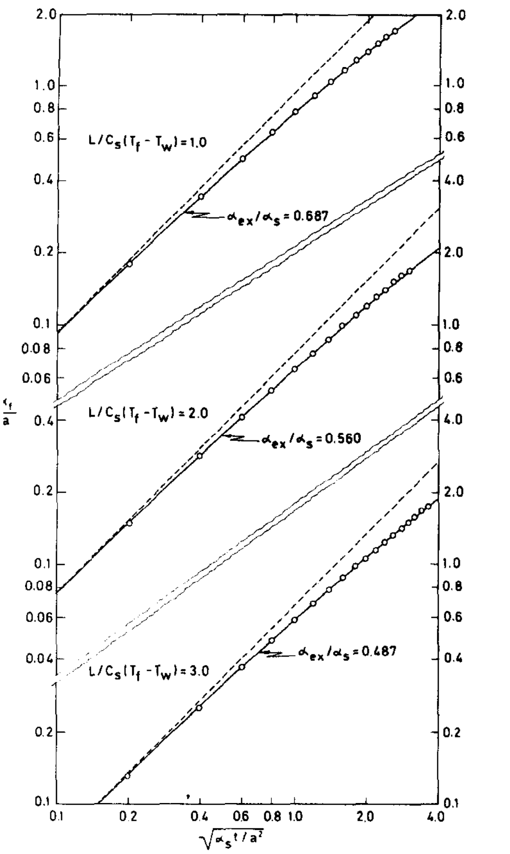

FIG. 1. Freezing-front location outside a cylinder. ——, Proposed approximation; ------, Flat plate; ○, Numerical solution (7): $k_s(T_f - T_w)/(k_l(T_0 - T_f) = 0.5$ and $\alpha_s/\alpha_l = 1.0$.

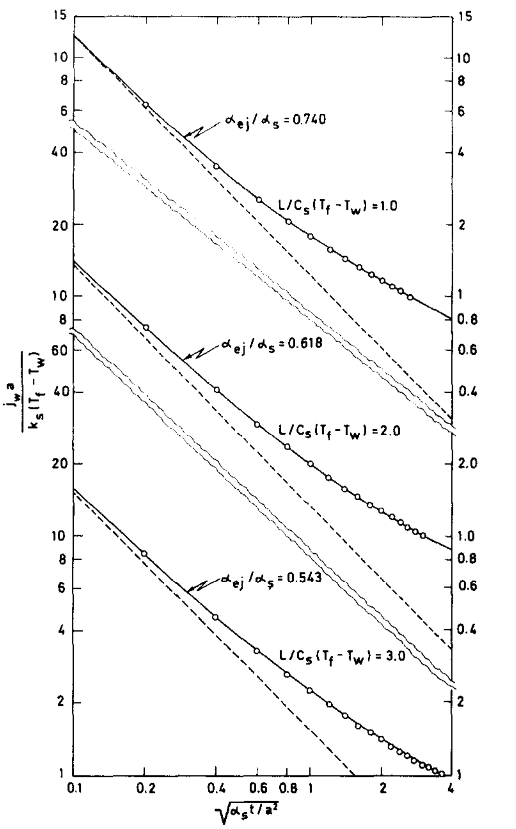

FIG. 2. Heat flux density at outside surface of a cylinder. ——, Proposed approximation; ------, Flat plate; ○, Numerical solution (7): $k_s(T_f - T_w)/k_l(T_0 - T_f) = 0.5$ and $\alpha_s/\alpha_l = 1.0$.

Approximate solution

Comparison with equation (5) indicates that equation (7) would give the exact solution for a finite latent heat if

$$k\rho c = (k\rho c)_s[(T_f - T_w)/(T_0 - T_w)\,\mathrm{erf}\{\lambda\}]^2. \quad (9)$$

This behavior can be forced by letting $k = k_s$ and taking the following effective value for the thermal diffusivity:

$$\alpha_{ej} = \alpha_s[(T_0 - T_w)\,\mathrm{erf}\{\lambda\}/(T_f - T_w)]^2. \quad (10)$$

This procedure can also be interpreted as choosing an effective heat capacity:

$$C_{ej} = C_s[(T_f - T_w)/(T_0 - T_w)\,\mathrm{erf}\{\lambda\}]^2 \quad (11)$$

and letting $k/\rho = (k/\rho)_s$.

The temperature field obtained by the use of α_{ej} is somewhat in error. However, the exact value of x_f can be obtained from equation (8) for $T_w \neq T_f$ by choosing

$$\alpha_{ex} = \alpha_s[\lambda/\mathrm{erf}^{-1}\{(T_f - T_w)/(T_0 - T_w)\}]^2 \quad (12)$$

α_{ex} does not, of course, give the exact heat flux when used with equation (7).

It is proposed to use these effective properties to approximate the behavior in other geometries and for other boundary conditions. The use of these effective properties permits representation of the non-linear problem of freezing by the linear model of pure conduction, for which a multitude of successful analytical and numerical methods have been developed. The choice of α_{ej} or α_{ex} depends on whether accuracy in the heat flux density or in the freezing-front location is of greater interest. If both quantities are important, both models can be used.

An analogous model can be developed based on the solutions for freezing with negligible sensible heat. However, no great advantage accrues relative to the model based on negligible latent heat and an equivalent body of methods and solutions does not exist.

EVALUATION OF MODEL

The proposed model can be demonstrated and tested by comparison of the results with the exact, numerical solutions for freezing for the following conditions.

Freezing outside a long cylinder

Tien and Churchill [7] obtained a solution for freezing outside a cylinder, under the same conditions as above for a flat plate, by numerical integration. The approximations obtained by using the above effective diffusivities with the analytical solution ([5], pp. 334–338 and [8]) for pure conduction are compared with this numerical solution for several parametric values in Figs. 1 and 2. The flat plate solution, which is an asymptote for time approaching zero, is included. Excellent agreement is apparent.

Freezing in a corner

Jiji *et al.* [9], Lazaridis [10] and Rathjen and Jiji [11] have carried out numerical calculations for freezing in a quarter-space, with both surfaces at the same uniform temperature. In Fig. 3 these calculated values for the freezing-front location in the plane of symmetry are compared with the predicted values using the analytical solution for pure conduction ([5], p. 171) and the effective diffusivity from equation (12). The agreement is reasonably good over the wide range of conditions.

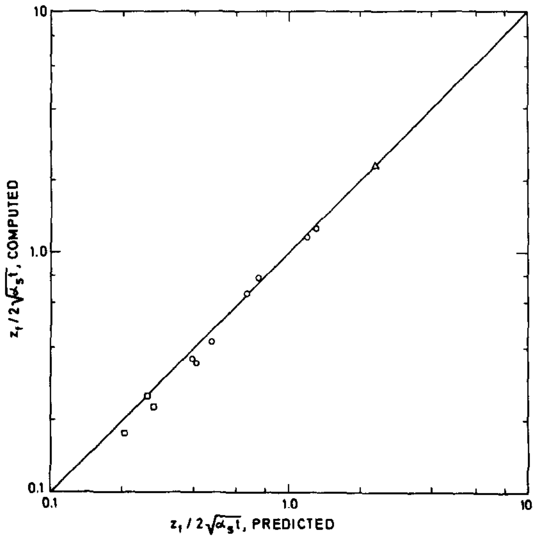

FIG. 3. Freezing-front location along the plane of symmetry in a square corner.

Symbols Authors	$\frac{k_l(T_0-T_f)}{k_s(T_f-T_w)}$	$\frac{\alpha_s}{\alpha_l}$	$\frac{L}{C_s(T_f-T_w)}$
□ Jiji *et al.* [9]	1.35	9.2	33.9
	0.553	9.2	22.4
	0.250	9.2	19.6
△ Lazaridis [10]	0.257	1.111	0.50
○ Rathjen and Jiji [12]	0.5, 2.00	1.0	0.1–10

CONCLUSIONS

Solutions for pure conduction with the effective diffusivities defined by equations (10) and (12) were found to agree closely with exact solutions for the heat flux density and the freezing front location, respectively. Similar accuracy is to be expected for the heat flux density and phase-front location in any geometry and with any boundary conditions, except that the freezing-front location cannot be calculated from equation (8) if $T_0 - T_f$. Corresponding approximations for the cases of convection in the liquid phase, a freezing range, moisture migration in wet soil, etc., can readily be formulated. The detailed temperature fields can also be calculated using these approximations, with equation (10) expected to give a better representation for the region near the surface and equation (12) near the freezing front.

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PERTURBATION SOLUTION FOR CONVECTIVE FIN WITH INTERNAL HEAT GENERATION AND TEMPERATURE DEPENDENT THERMAL CONDUCTIVITY

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NOMENCLATURE

- Bi, Biot number, $= hw/2k_a$;
- E, fin effectiveness, $= Q/hw(T_b - T_a)$;
- G, generation number, $= qw/2h(T_b - T_a)$;
- h, heat-transfer coefficient;
- k, thermal conductivity;
- L, fin length;
- N, fin parameter, $= \left(\frac{2h}{k_a w}\right)^{1/2} L$;
- q, volumetric rate of heat generation;
- Q, heat-transfer rate;
- T, temperature;
- w, fin thickness;
- x, axial distance measured from fin tip;
- X, dimensionless axial distance, $= x/L$.

Greek symbols

- β, slope of thermal conductivity–temperature curve divided by intercept k_a;
- θ, dimensionless temperature;
- ε, thermal conductivity parameter; $(k_b - k_a)/k_a = \beta(T_b - T_a)$.

Subscripts

- a, environment;
- b, fin base.

INTRODUCTION

IN FIN literature one finds several papers focussing attention on the effect of internal heat generation on the performance of convective fins. For example, Minkler and Rouleau [1] studied rectangular and triangular fins with uniform internal